


Matura Examinations 2022 – Mathematics

Classes: 4Be, 4LW

Teachers: BtT, ScR

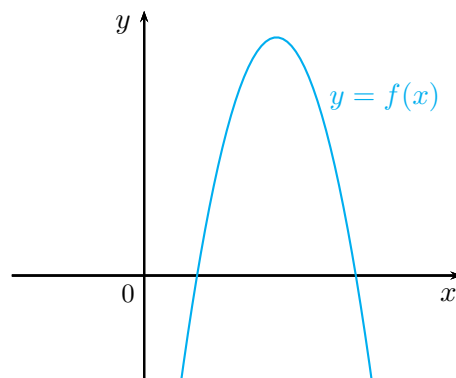
- Note: You have four hours to complete the examination.
Begin each question on a new sheet of paper.
- Permitted materials: TI-*nspire* CX calculator (in 'press-to-test' mode)
The *Fundamentum Mathematics and Physics*, without notes
English-German dictionary


All questions labelled with the symbol  are to be solved **by hand**. For these questions, only the basic functions of your calculator are permitted. To attain full marks in these questions, you should not use commands such as dotP, nSolve, polyRoots or the numerical calculation of derivatives or integrals.

In general, the graphics window of your calculator should only be used to visualise the graphs of functions.


Question 1: Analysis (11 marks)

Shown below is the graph of the function $f(x) = -x^2 + 10x - 16$.



-  (a) Find the coordinates of the x -intercepts and the local maximum of the graph $y = f(x)$. (2.5 P.)
- (b) The three points in part (a) form a triangle. Calculate the length of its perimeter. (1.5 P.)
- (c) The graph of f also crosses the y -axis, although the y -intercept cannot be seen in the figure above. What angle does the graph form with the y -axis at this point? (1.5 P.)

Consider the region enclosed between the graph of f and the x -axis. Suppose we inscribe a rectangle in this region, with two vertices lying on the x -axis and the other two vertices on the graph of f . Denote by u the length of the rectangle's horizontal side, and by v the length of its vertical side.

- (d) Make a sketch of the situation. (0.5 P.)
- (e) Explain why the length of the rectangle's vertical side satisfies the formula $v = f(5 + \frac{u}{2})$. (1 P.)
-  (f) Calculate the maximal possible area of such a rectangle. (4 P.)

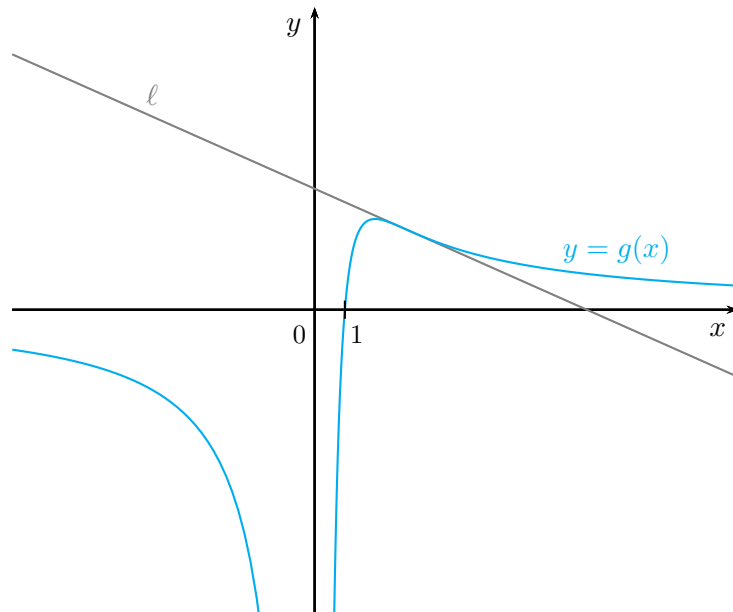
Question 2: Analysis (12 marks)

Note: The two parts (a) and (b) can be solved independently of each other.

(a) Shown below are the graph of the rational function

$$g(x) = \frac{12x - 12}{x^2}$$

and the straight line ℓ with equation $y = -\frac{4}{9}x + 4$.



- i.** Find the equations of all the asymptotes of the graph of g . Justify your answers. (1.5 P.)
- ii.** Find the coordinates of the point of intersection of the graph $y = g(x)$ with the line ℓ . Show using relevant calculations that the line ℓ is tangent to the graph at this point. (2.5 P.)
- iii.** Calculate the area of the bounded region which is enclosed between the graph $y = g(x)$, the tangent ℓ and the x -axis. Give an exact answer. (4 P.)

(b) Consider the rational function

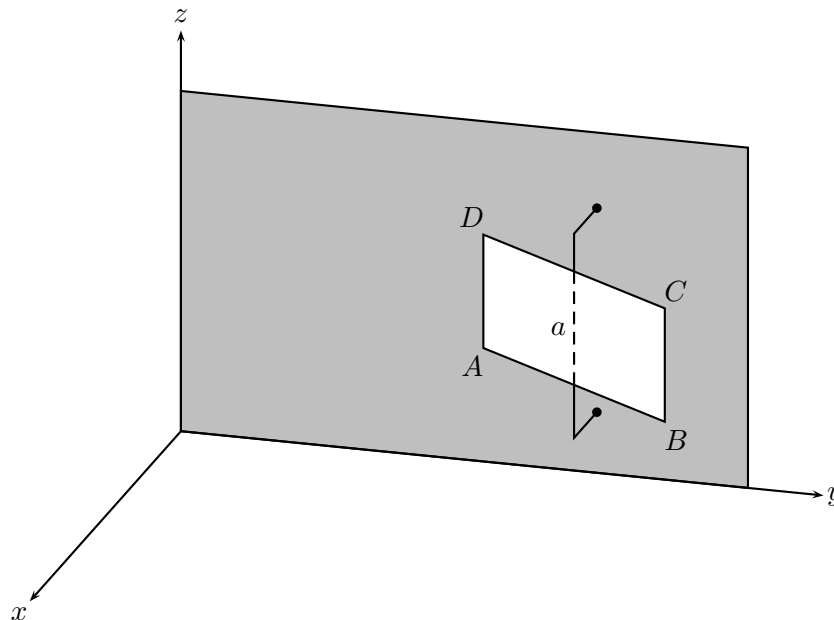
$$h(x) = \frac{ax^2 + b}{x + c}$$

Determine the parameters a , b and c so that the graph of h has a vertical asymptote at $x = 2$ and a stationary point at $P(6, 3)$. (4 P.)

Question 3: Vector Geometry (12 marks)

In the model of a conference room, one of the walls is represented by the yz -coordinate plane. Attached to this wall is a rectangular screen $ABCD$, which can be rotated around the vertical axis a (see below). The centre of the screen lies on this axis.

The coordinates of three of the screen's corners are given: $A(0, 4, 1.5)$, $B(1.5, 7, 1.5)$ and $C(1.5, 7, 3)$.



- Calculate the coordinates of the fourth corner D . (0.5 P.)
- Calculate the area of the screen $ABCD$. (2 P.)
- Denote by Γ the plane containing the screen $ABCD$. Find a Cartesian equation for Γ , and show that it is equivalent to the equation $2x - y + 4 = 0$. (2.5 P.)
- Note that the screen is currently positioned so that its left side \overline{AD} is touching the wall. Calculate the angle formed between the screen and the wall. (2 P.)

Suppose now that during a presentation, a laser pointer is used at the point $L(4, 3, 2)$.

- The laser pointer emits a beam of light which meets the plane Γ at the point P . Assume that the laser pointer is held so that the distance between L and P is as short as possible. Find the coordinates of the point P . (2 P.)
- Verify whether the point P lies on the screen $ABCD$. (0.5 P.)
- Find a vector equation for the straight line which contains the rotational axis a of the screen. (1.5 P.)
- Suppose the screen is rotated around the axis a , so that the right side \overline{BC} is touching the wall. The new Cartesian equation for the plane containing the screen is $4x + 2y + k = 0$. Determine the value of k . (1 P.)

Question 4: Probability (12 Punkte)

Note: Parts (a), (b) and (c) can be solved independently of each other.

(a) As a health and safety precaution, 15 people are chosen at random from a group of 24 travellers to be tested for the coronavirus using a “rapid antigen test”.

i. How many different ways are there for choosing the 15 test subjects? (1 P.)

The group of travellers consists of 17 adults and 7 children.

ii. What is the probability that all seven children are chosen for testing? (1.5 P.)

iii. What is the probability that at least five children are chosen for testing? (1.5 P.)

(b) Suppose that on average, 6 out of every 1000 rapid antigen tests are damaged and cannot be used. The tests are sold in boxes containing 150 tests each.

i. What is the probability that among all the tests in a single box, exactly two are damaged? (1 P.)

ii. What is the probability that among all the tests in a single box, more than four are damaged? (1.5 P.)

iii. Find the smallest number of boxes which would have to be examined for the probability of finding at least one damaged test to be higher than 98%. (2 P.)

(c) For the remainder of this question, we assume that only undamaged rapid antigen tests are used. These tests have a *sensitivity* of 95% and a *specificity* of 98%. This means that

- for a person who really does have the coronavirus, the probability of the test showing positive is 95%, whereas
- for a person who does not have the coronavirus, the probability of the test showing negative is 98%.

i. Suppose that in Village X, every tenth person is infected with the coronavirus.

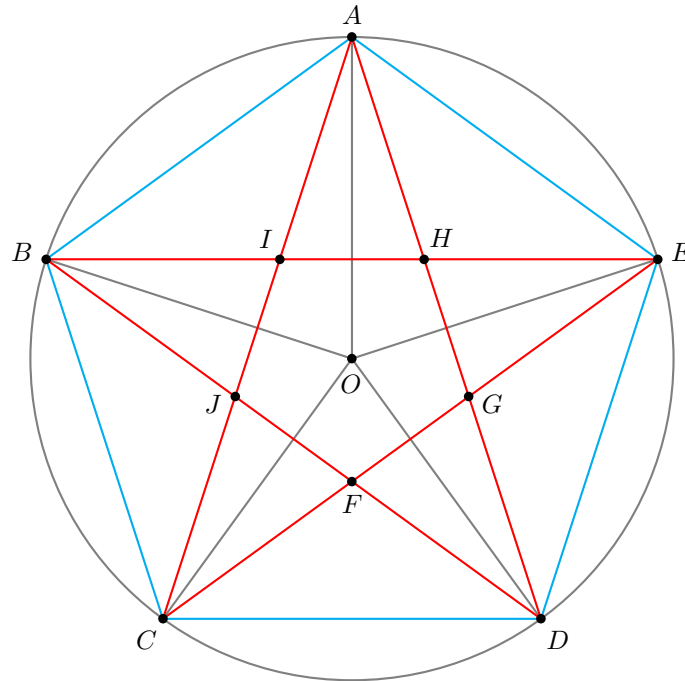
A randomly chosen inhabitant of Village X tests positive for the coronavirus. Calculate the probability that this person really does have the coronavirus. (1.5 P.)

ii. In another Village Y, all 1000 inhabitants are tested using the rapid antigen tests described above. In total, 40 inhabitants receive a positive test result.

Estimate the number of inhabitants of Village Y who will have received a negative test result despite being infected with the coronavirus. (2 P.)

Question 5.1: Trigonometry (7 marks)

The picture below shows a regular pentagon $ABCDE$ inscribed in a circle. The perimeter of the pentagon is 25 cm long.



- Calculate the angle $\angle AOB$. (0.5 P.)
- Calculate the radius of the circle. (1 P.)
- Calculate the length of the diagonal \overline{AC} . (1.5 P.)
- The five diagonals of the original pentagon $ABCDE$ form a new, smaller pentagon $FGHIJ$. Calculate the perimeter of $FGHIJ$.

Hint: Note that the triangles ACB and BAI are similar. They are both isosceles and have the same base angle $\angle BAC = \angle BAI$. (2 P.)

If you weren't able to solve part (d), assume in the following that the perimeter of $FGHIJ$ is equal to 9.6 cm.

- Denote by p_0 the perimeter of the original pentagon $ABCDE$, and by p_1 the perimeter of the new pentagon $FGHIJ$. The five diagonals of $FGHIJ$ form another new pentagon; denote its perimeter by p_2 . We can continue the process indefinitely, forming pentagons with perimeters equal to p_3, p_4, p_5 , and so on.

Calculate p_2 and p_{20} . Give your answer for p_{20} in nanometres. ($1 \text{ nm} = 10^{-9} \text{ m}$)

(2 P.)

Question 5.2: Logarithms (6 marks)

Search engines like Google and Yahoo use **page rank algorithms** to classify websites in the internet, so that they can list the most relevant pages first. The higher the page rank of a website, the more often it is visited.

Suppose a search engine uses the following formula to calculate page ranks: for a website with x visitors per month, the page rank y is given by

$$y = 1.5 \log \left(\frac{x}{250} \right)$$

where \log denotes the base 10 logarithm function.

✎(a) The website `www.page1.ch` has page rank equal to 6. How often is it visited per month? (1.5 P.)

(b) The website `www.page2.ch` has one hundred times more visitors per month than the website `www.page3.ch`. How much higher is its page rank? (2 P.)

✎(c) Another search engine uses a different formula to calculate page ranks: for a website with x visitors per month, the page rank y is given by

$$y = a \log \left(\frac{x}{b} \right)$$

Find the values of a and b so that a website with 160000 monthly visitors has page rank 4, whereas a website with 200 million monthly visitors has page rank 8. (2.5 P.)