Matura Examinations 2021 – Mathematics (GLF)

Klassen: 4B, 4Be, 4BW, 4GLW, 4KSW, 4M, 4S, 4SI, 4W, 4Z
Lehrpersonen: BoJ, BtT, HnR, HrP, KiA, MoM, PeM, RoL, SuF

Remarks:	The duration of this exam is 4 hours.
	Start each question on a fresh sheet of paper!
Material:	Calculator TI-*n*spire CX in Press-to-Test-Modus
	Fundamentum Mathematik und Physik, without any hand-written notes. English-German dictionary

In questions which are to be solved by hand, only the basic functions of your calculator are permitted. To attain full marks in these questions, you should not use commands such as `dotP()`, `nSolve()`, `polyRoots()` or the numerical calculation of derivatives or integrals. For each part-question that requires a by hand solution, the symbol ✍ has been drawn. In general, the graphics window of your calculator should only be used to visualise the graphs of functions.

Question 1: Vector Geometry

The three points $A(5|8|6)$, $B(7|9|8)$, $C(9|-5|13)$, together with the plane $E_1 : 2x - 3y - z = 4$ are given.

✍ (a) Show, using appropriate calculations, that the triangle ABC has a $90°$ angle at vertex B. (1.5 P.)

(b) The shape $ABCD$ is a rectangle. Find the coordinates of the point D. (1.5 P.)

(c) Find a Cartesian equation for the plane E_2, that contains the rectangle $ABCD$. (2 P.)

If you are not able to solve part (c), you should use the Cartesian equation $E_2 : -22x + 4y + 20z = 42$ for E_2, instead.

(d) Calculate the angle between E_1 and E_2. (1.5 P.)

(e) A source of light is found at point P. The light shines straight out in all directions. The effect of this light causes the rectangle $ABCD$ to cast a shadow on the xy-plane. The shadow shape itself has four vertices: $A'(3|4|0)$, $B'\left(5\left|\frac{7}{2}\right|0\right)$, C' and D'.

Calculate the coordinates of the light source. (3 P.)

If you are unable to find the position of the light source, P, you should calculate an answer to part (f) using the substitute coordinates $P'(-1|-4|-12)$.

(f) A pyramid can be constructed with base $ABCD$ and summit-point (*Spitze*) P. By applying the Triple-Product Formula $\left(V = \frac{1}{3}(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AP}\right)$, the volume of this pyramid can be quickly calculated. However, you are required to find the volume of this pyramid using a **different** method. (2.5 P.)

Question 2: Calculus

The polynomial function f is given as:

$$f(x) = 2x^3 - 165x^2 + 2700x$$

The graph of f is given the name G_f.

(a) Find the x-coordinate(s) of any local maxima or local minima on G_f.

Note: It is not necessary to identify their type. (1 P.)

(b) Find the x-coordinate(s) of any point(s) of inflexion on G_f. (1 P.)

(c) A shovel (Schaufel) can be constructed by the removal of two identical squares (side-length $= x$ cm) from a rectangular sheet of metal (*Blech*). The sheet measures 60 cm $\times 45$ cm. The diagram below shows how the metal sheet must be folded in order to form the shovel.
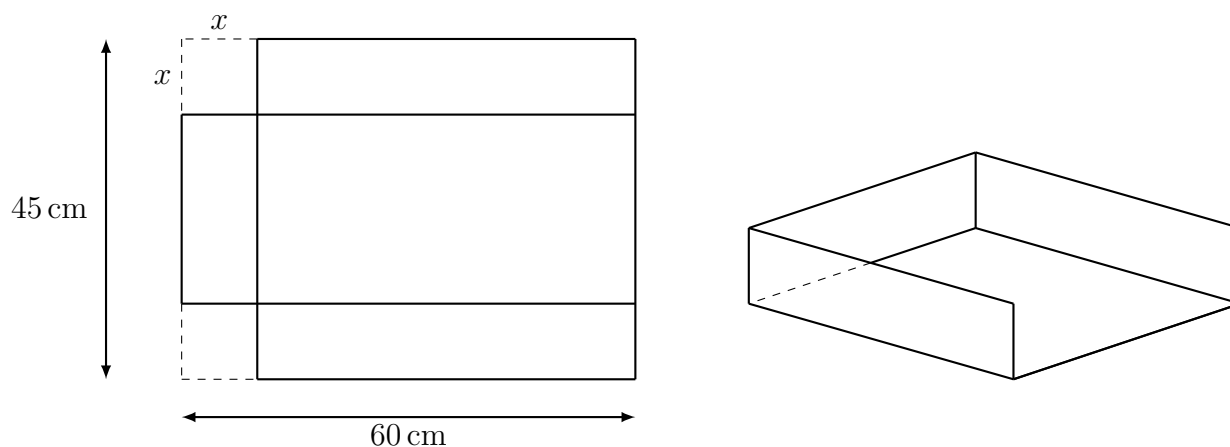
(i) Show that function f describes the volume (*Fassungsvermögen*) of this shovel in cm^3 in terms of the side-length x. (1.5 P.)

(ii) State a clear interval for the x-values which could be used in (c)(i). (0.5 P.)

(iii) Calculate the dimensions (length, width and height) required to produce a shovel with maximum volume, and state the value of this maximum volume. (2 P.)

(d) As an introductory exercise to the topic of Optimisation, different shovels are distributed to a class of students - each shovel having been constructed using the same technique as in part (c) of the question, using a 60 cm $\times 45$ cm rectangle of a stiff, but bendable, metal sheet (*Blech*).

One square centimetre of this metal sheet has a mass of 2 g.

The students are then asked to measure and record: the *height* in centimetres, the *volume* in litres and the *weight* in kilograms of their individual shovel. All measurements are recorded accurate to three decimal places.

(i) Petra's shovel has a recorded mass of 5.076 kg. What was her recorded volume measurement? (2 P.)

(ii) **Use of a calculator is recommended for this part-question.**
The shovels of Mark and Céline have the same volume, but Mark's shovel is 9 cm taller than Céline's. What is the volume of Céline's shovel? (2 P.)

Question 3: Calculus

The two functions g and h are given as:

$$g(x) = \left(3x - x^2\right) \cdot \mathrm{e}^x \quad \text{und} \quad h(x) = 2x^3 - 12x^2 + 18x \quad \text{for } x \in \mathbb{R},$$

where e is Euler's number.

(a) Find all zeros of function g. (1.5 P.)

(b) Show that $G(x) = \left(5x - 5 - x^2\right) \cdot \mathrm{e}^x$ is an integral function of g. (1.5 P.)

(c) Calculate the **exact** area of the following bounded regions lying in the 1st quadrant. :

   (i) between the graph of function h and the x-axis. (3 P.)

   (ii) between the graph of function g and the x-axis. (1 P.)

(d) Calculate an **approximate** value for the **total area** bounded by the graphs of functions h and g. (1.5 P.)

(e) A third function, w, is defined as follows:

$$w(x) = k \cdot h(x) \text{ , where } k \text{ is a real constant value.}$$

Find the value of k for which the point of inflexion on the graph of function w also lies on the graph of the function g. (3 P.)

Question 4: Stochastics

Bilbo the Hobbit and Gandalf the Wizard are expecting visitors, 9 male and 4 female Dwarves: the leaders of their people.

1. The 13 Dwarves arrive one after the other.

    (a) What is the probability that the first three visitors to arrive are all female Dwarves?   (1 P.)

    (b) What is the probability that among the first three visitors to arrive there are at least two males?   (1 P.)

2. After waiting for a long time, Gandalf gets bored and he claims that he can guess the names of the first four guests (without the use of magic). Gandalf knows the names of all 13 Dwarves, of course.

    (a) What is the probability that Gandalf guesses all four names in the right sequence by chance? (1 P.)

    (b) What is the probability that Gandalf guesses all four names but not necessarily in the right sequence?   (1 P.)

3. After an intensive campaign, the percentage of smokers amongst Dwarves has fallen to 55%. Gandalf and Bilbo know nothing about the habits of their guests.

    (a) What is the probability that none of the thirteen guests are smokers?   (1 P.)

    (b) What is the probability that there are at least two non-smokers?   (1 P.)

    ✎ (c) What is the minimum number of Dwarves that Bilbo and Gandalf would have to invite to reach a probability of at least 99.99% that there are one or more smokers amongst them? (2.5 P.)

    (d) 60% of all Dwarves are male. 64% of all female Dwarves smoke. What is the probability (rounded to the nearest percentage point) that a randomly selected smoking Dwarf is male? (1.5 P.)

✎ 4. The probability p, that a Dwarf has curly hair is unknown. Now, Gandalf has the revelation that the probability that exactly one of the two Dwarves arriving first has curly hair is exactly 45.5%. What is the value for p if Gandalf is right?

Draw a fully labeled tree diagram for the problem, create an equation in term of p, and solve it. (2 P.)

Question 5.1: Trigonometry

Calculate the values for $\overline{BC}$, α and $\overline{CD}$ from the following diagram: (3 P.)
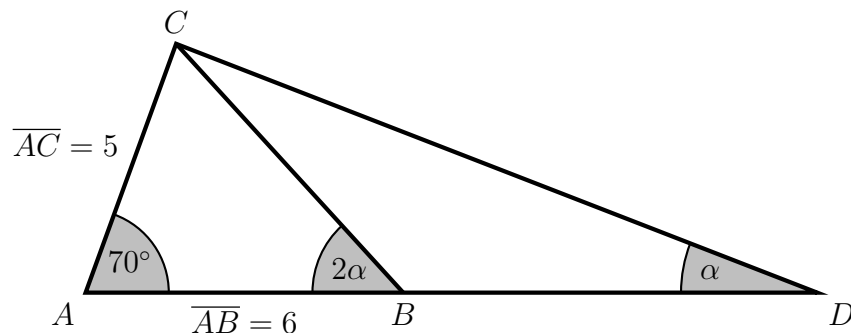
Question 5.2: Beer Foam and Beer Glasses

Researchers have shown that the foam on the top of a beer decays exponentially, and that the **decay constant** changes with the type of beer. The height, in cm, of foam on a **lager beer** held in a cylindrical container obeys the following formula:

$$h(t) = h_0 \cdot e^{-0.2t}$$

where h_0 represents the intial height of foam, t the time in seconds that have passed and e represents Euler's Constant.

(a) Using a cylindrical container, the decay of the foam of a lager beer is investigated. Immediately after pouring in the lager beer, a $10\,cm$. high *head of foam* (*Schaumkrone*) is observed.

How high will this *head of foam* be after 8 seconds?

Give your answer accurate to the nearest centimetre. (1 P.)

(b) In my local pub, lager beer is served with a $3\,cm$ high *head of foam*. How long must I wait if I will only start to drink when the *head of foam* has shrunk to $1\,cm$ in height? Give here, the **exact** answer. (1.5 P.)

(c) The height of a *head of foam* of a **wheaty beer** (Weissbier) obeys the following formula:

$$w(t) = w_0 \cdot e^{-0.3t}$$

A lager beer and a wheaty beer are poured simultaneously (*gleichzeitig*) into separate cylindrical containers. The lager beer has a *head of foam* of $5\,cm$ and the wheaty beer has a *head of foam* of $10\,cm$.

When will the two beers have *heads of foam* of the same height?

Give your answer accurate to two decimal places. (1.5 P.)

(d) «*Belgian beer tastes best in a Kelch!*» - At least, this is the claim made by the firm *Glass4you*. The perfect Kelch-shape can be constructed using the function f with the following equation:

$$f(x) = \frac{1}{4}\sqrt{x}(12 - x)$$

When the curve is rotated around the x-axis interval $[0, 8]$ ($0 \le x \le 8$), the perfect glass shape is produced (see diagram). Note: One unit equals one centimetre.
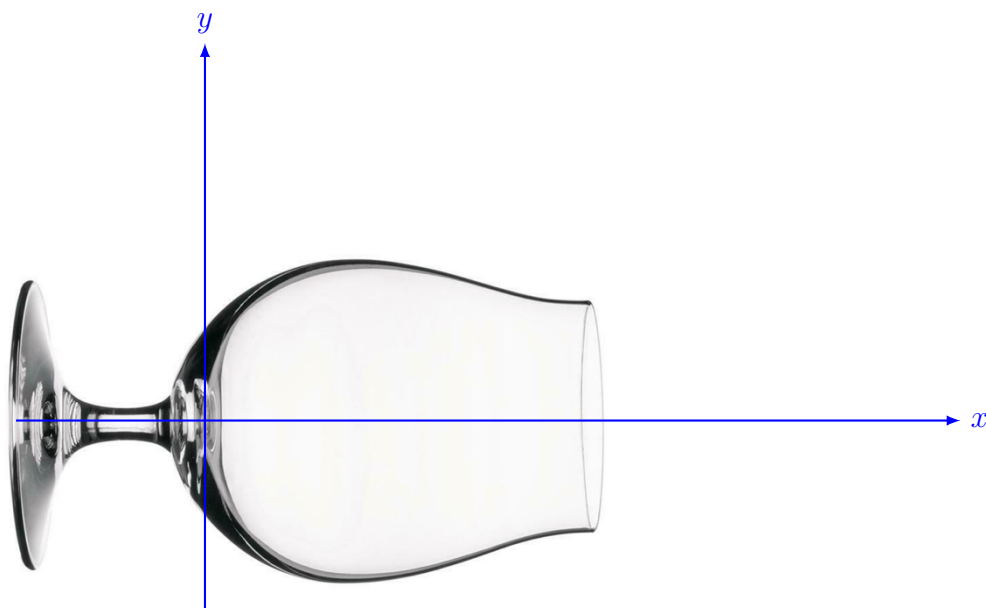
How many decilitres (dl) of beer can this Kelch hold?   (2.5 P.)