Matura Examination 2019 – Mathematics

Classes: 4Be, 4BL, 4We
Teachers: BtT, PrG, ScR

Note:	You have four hours to complete the examination.
	Begin each question on a new sheet of paper.
Permitted materials:	TI-*n*spire CX calculator (in 'press-to-test' mode)
	The *Fundamentum Mathematics and Physics*, without notes
	English-German dictionary

In questions which are to be solved **by hand**, only the basic functions of your calculator are permitted. To attain full marks in these questions you should not use commands such as dotP, nSolve, polyRoots or the numerical calculation of derivatives or integrals.

In general, the graphics window of your calculator should only be used to visualise the graphs of functions.

Question 1: Vector Geometry

Consider the points $A(-3,-1,7)$, $B(7,4,-3)$ and $C(-3,14,-8)$.

(a) Show **by hand** using appropriate calculations that the triangle ABC is isosceles[1] and has a right angle at the vertex B. (1.5 P.)

(b) Find **by hand** the coordinates of the point D on the y-axis which lies at the same distance from the point A as it does from the point C. (2 P.)

(c) Find a Cartesian equation for the plane $\mathcal{E}_1$ which contains the three points A, B and C. (1.5 P.)

If you are not able to solve part (c), you should use the Cartesian equation $x+2y+2z-18=0$ for $\mathcal{E}_1$ instead.

(d) Show that the point $F(-\frac{5}{2}, 12, \frac{19}{2})$ does not lie in the plane $\mathcal{E}_1$. Calculate the coordinates of the point F' obtained by reflecting F in the plane $\mathcal{E}_1$. (3 P.)

Denote by $\mathcal{E}_2$ the plane with Cartesian equation $2x-5z+3=0$.

(e) How can we tell that the two planes $\mathcal{E}_1$ and $\mathcal{E}_2$ intersect? Calculate the angle between them. (2 P.)

(f) Find a vector equation for a straight line which intersects neither $\mathcal{E}_1$ nor $\mathcal{E}_2$. (2 P.)

[1] isosceles = *gleichschenklig*

Question 2.1: Calculus

Denote by f the function

$$f(x) = x^3 - 6x^2 + 9x$$

and by ℓ the straight line with equation $y = x$.

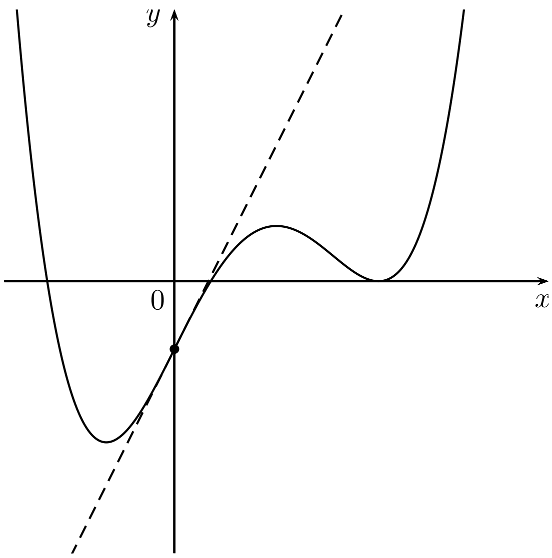
(a) Determine **by hand** the zeroes of the function f, as well as the coordinates of any local maxima, local minima and points of inflection. (4 P.)

(b) Calculate the points at which the straight line ℓ intersects the graph of f. (0.5 P.)

(c) Calculate **by hand** the total area enclosed between ℓ and the graph of f. (2.5 P.)

Question 2.2: Calculus

Shown below is the graph of a polynomial function g of degree four, and the dashed line with equation $y = 2x - 2$. The two graphs intersect at a point on the y-axis.



The dashed line is tangent to the graph $y = g(x)$ at a point of inflection of g. The function g has local extrema at $x = -2$ and $x = 3$.

Find an expression for $g(x)$. You should solve the resulting system of equations **by hand**. (5 P.)

Question 3: Calculus

Consider the function

$$h(x) = \mathrm{e}^{1-x}$$

(a) Where does the graph of h intersect the y-axis? (0.5 P.)

(b) As the value of x increases, the graph of h gets closer and closer to the x-axis. For which values of x is it true that $h(x)$ is smaller than 0.001? Solve this question **by hand** and give an exact expression for the answer. (1.5 P.)

(c) Find an equation for the tangent to the graph $y = h(x)$ at the point $x = -1$. (2 P.)

(d) Show that the function $H(x) = 2 - \mathrm{e}^{1-x}$ is an anti-derivative of h. (0.5 P.)

(e) Calculate the **exact** area of the region in the first quadrant which lies between the graph of h and the coordinate axes. (1.5 P.)

Now consider the quadratic function

$$p(x) = 2x^2 - 1$$

(f) Calculate the angle formed by the graphs of h and p at their point of intersection. (2 P.)

(g) Which points on the graph of p lie closest to the origin? To obtain full marks, you should solve this question **by hand**. (4 P.)

Question 4: Probability

George, Charlotte and Louis work in a clothing factory sewing shirts. The quality of their work is regularly inspected. Of the shirts sewn by George, 95% fulfill the factory's quality standards; the corresponding figures for Charlotte and Louis are 90% and 85%, respectively.

(a) Calculate the probability that of 15 shirts sewn by Louis, exactly 12 meet the factory's quality standards. (1 P.)

(b) The factory managers suspect that Louis' work is not of a very high standard. They order a separate examination of the shirts he sews. What is the minimum number of shirts that should be inspected so that the probability of discovering at least one faulty shirt is greater than 99%? (2 P.)

On a particular morning, George sews 20 shirts, Charlotte 25 shirts and Louis 15 shirts. All 60 shirts are clearly distinguishable[2] from one another. The factory inspector chooses a total of ten of these shirts at random, without knowing which shirt was sewn by which employee.

(c) In how many different ways can the inspector choose the ten shirts? (1 P.)

(d) What is the probability that none of the ten shirts chosen for inspection were sewn by Louis? (1.5 P.)

As it happens, of the ten shirts chosen for inspection, three were sewn by George, five by Charlotte and two by Louis. The inspector arranges the ten shirts in a random order and begins the quality control.

(e) What is the probability that the first shirt examined by the inspector passes the quality control? (2 P.)

(f) What is the probability that the three shirts sewn by George are examined last? (1.5 P.)

(g) What is the probability that all ten shirts pass the quality control? (1.5 P.)

(h) Suppose that the first shirt passes the quality control. What is the probability that it was sewn by Louis? (1.5 P.)

[2]distinguishable = *unterscheidbar*

Question 5.1: Combinatorics

The student association (SA) of Gymnasium Liestal wants to paint a colourful version of the school's old logo above the main entrance to the school.

gymnasiumliestal

The SA decides on the following rules:

- Each of the 16 letters should be painted with a single colour (ie., no stripes, dots or other patterns are allowed).
- Only colours from the SA's own paintbox may be used. The paintbox contains 19 different colours.

(a) In how many different ways can the logo be painted according to these rules? (1 P.)

(b) In how many different ways can the logo be painted if, in addition to the rules listed above, no colour may be used more than once? (0.5 P.)

The SA presents the school principals with 24 different designs for the colourful logo. The best design will be chosen by a committee of 10 teachers selected from the school's staff[3]. The school's staff consists of 81 women and 106 men, and includes the five school principals (two women and three men).

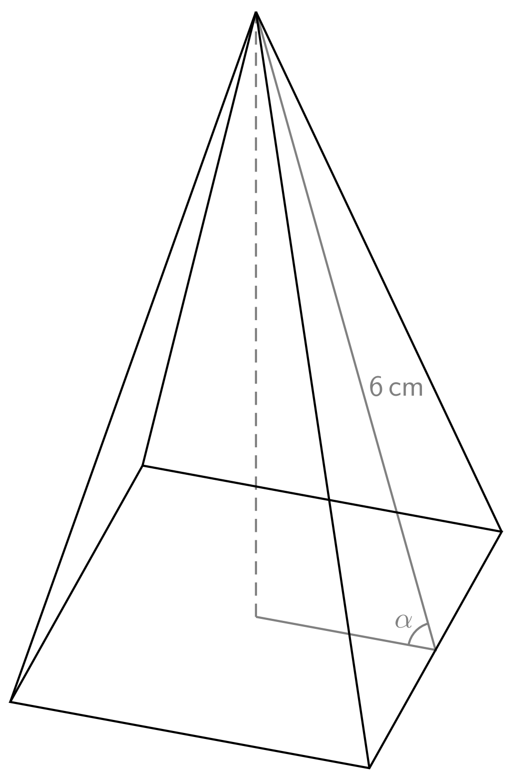
(c) How many different committees could be selected? (1 P.)

(d) How many different committees could be selected which contain all five school principals and the same number of men as women? (2 P.)

[3] staff = Lehrerkollegium

Question 5.2: Optimisation

Suppose we want to construct a pyramid like the one shown below. The base should be square and the sides should be isosceles triangles, each of height 6 cm. Denote by α the angle formed between the base and the sides of the pyramid.



Note: in the following, we use $\sin^3(\alpha)$ *to denote* $(\sin(\alpha))^3$.

(a) Show that for any given angle α, the volume of the pyramid can be expressed by the formula

$$V(\alpha) = 288\left(\sin(\alpha) - \sin^3(\alpha)\right)$$

Note: to obtain this formula, you will need to use the fact that $\sin^2(\alpha) + \cos^2(\alpha) = 1$ *for any angle* α. (3.5 P.)

(b) Calculate **by hand** the angle for which the pyramid has the greatest volume.

Note: you are not required to verify your answer with the second derivative test. (4 P.)