gymnasium liestal

Matura Examination 2018 – Mathematics

Classes: 4Bb, 4LW (Profile L), 4Wb

Note:	You have four hours to complete the examination.
	Begin each question on a new sheet of paper.
Permitted materials:	TI-*n*spire CX calculator (in 'press-to-test' mode)
	The *Fundamentum Mathematics and Physics*, without notes
	English-German dictionary

In questions which are to be solved **by hand**, only the basic functions of your calculator are permitted. To attain full marks in these questions you should not use commands such as nSolve, polyRoots or the numerical calculation of derivatives or integrals.

In general, the graphics window of your calculator should only be used to visualise the graphs of functions.

Question 1: Calculus

Consider the polynomial function

$$f(x) = \tfrac{1}{9}x^3 - x^2 + 10$$

Let P denote the point of inflection of the graph of f.

(a) Calculate **by hand** the coordinates of the point P. (2.5 P.)

(b) Find an equation for the tangent to the graph of f at the point P. (1.5 P.)

(c) The parabola with equation $y = \frac{1}{2}x^2 + x - \frac{7}{2}$ intersects the graph of f at the point P. Calculate the angle between the two curves at this point. (2 P.)

(d) The parabola with equation $y = -x^2 + x + 10$ intersects the graph of f at the point P and at two other points. Calculate **by hand** the area of the region enclosed between this parabola and the graph of f. (3 P.)

(e) A certain polynomial function p of degree four has a graph which is symmetric in the y-axis, intersects the y-axis at $y = 10$ and is tangent to the graph of f at $x = 9$.

Find an expression for $p(x)$. (3 P.)

Question 2: Calculus

Consider the rational function

$$g(x) = \frac{20x - 40}{x^3}$$

(a) Calculate the zeroes of the function g and find the equations of the asymptotes of the graph of g. (2 P.)

(b) This question should be answered **by hand**.

Calculate the coordinates of the local extremum of the function g and determine whether this point is a local maximum or a local minimum. (3 P.)

(c) For each real number $k > 2$, we denote by A_k the area of the region in the first quadrant which is enclosed between the graph of g, the x-axis and the vertical line $x = k$.

Calculate the area A_k and determine the limit of A_k as k tends to ∞. (3 P.)

(d) Suppose that the point $P(u, v)$ lies on the graph of g, where $u > 2$. Consider the triangle with vertices $Z(2, 0)$, $Q(u, 0)$ and $P(u, v)$.

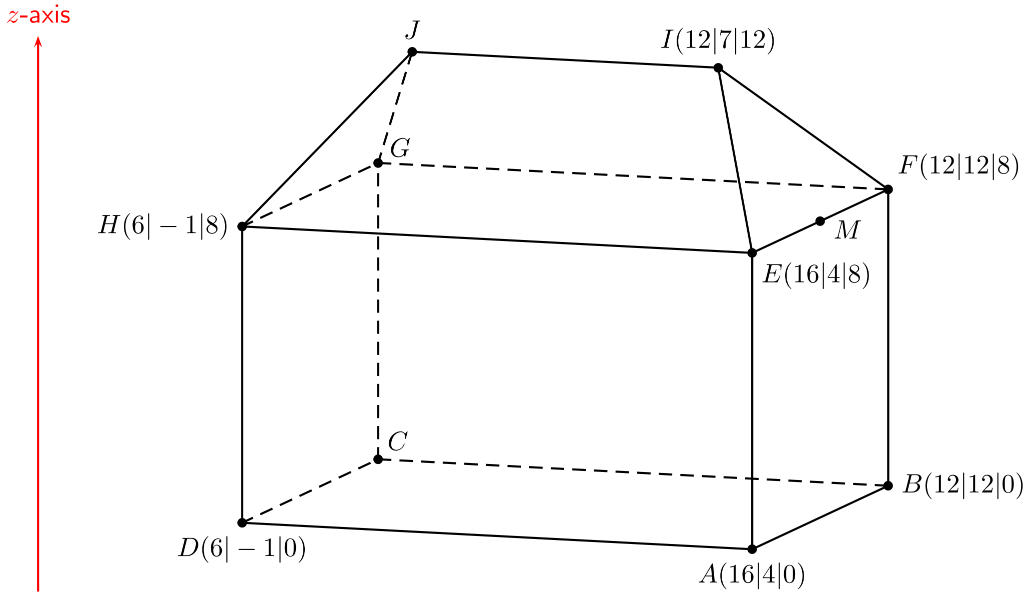
Make a sketch of the situation. Calculate **by hand** the value of u for which the area of the triangle ZQP is maximal. (4 P.)

Question 3: Vector geometry

cuboid	*Quader*
isosceles triangle	*gleichschenkliges Dreieck*
beam	*Balken*
neglect	*vernachlässigen*

The sketch below shows the outline of a wooden house. The main part of the house is a cuboid; the roof is a so-called 'hipped roof' consisting of two identical trapeziums and two identical, isosceles triangles. The ridge $\overline{JI}$ of the roof is horizontal. The floor of the house lies in the xy-coordinate plane.

For simplicity's sake, the thickness of the wooden beams of the house will be neglected.



(a) The point M is the midpoint of the beam $\overline{EF}$.

Find the coordinates of the points C, M and J. (3.5 P.)

(b) Find a Cartesian equation for the plane containing the trapezium $HEIJ$. (2 P.)

(c) Given that the triangle EIF lies in the plane $8x + 4y + 5z - 184 = 0$ and the trapezium $FGJI$ lies in the plane $-2x + 4y + 5z - 64 = 0$, calculate the angle formed by EIF and $FGJI$. (2 P.)

(d) To improve the stability of the roof, an extra beam is added between the point M and the beam $\overline{FI}$ so that the two beams meet each other at right-angles.

Find the coordinates of the point where the two beams meet. (2.5 P.)

(e) A bird flies above the ridge of the house along the straight line with vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 11 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

What is the closest distance the bird comes to the line JI? (2 P.)

Question 4: Probability

In springtime, Theresa plans to grow tulips on her balcony. She visits a shop with a very large supply of tulip bulbs. These bulbs are mixed together in small bags of five bulbs. Each bulb is indistinguishable in shape, size or colour from any other tulip bulb. On average, 50% of these bulbs produce red tulips, 30% produce yellow tulips and 20% white tulips.

(a) Theresa is considering buying one bag containing five bulbs. Calculate the probability

  i. that none of the bulbs in the bag produce a white tulip. (1 P.)

  ii. that exactly one of the five bulbs produces a red tulip. (1 P.)

  iii. that at least two of the bulbs produce a yellow tulip. (1.5 P.)

  iv. that all five bulbs produce tulips of the same colour. (1.5 P.)

(b) Upon reflection, Theresa changes her mind and decides to buy two bags of bulbs.

How many different pairs of bags could she choose from the 14 which are available in the shop? (1 P.)

(c) After planting and watering her ten bulbs, Theresa is rewarded by the appearance of seven red tulips, two yellow tulips and one white tulip. She plans to place them in a straight line on her balcony.

  i. In how many different colour arrangements can she place these ten tulips so that there is a red tulip at each end of the row? (1.5 P.)

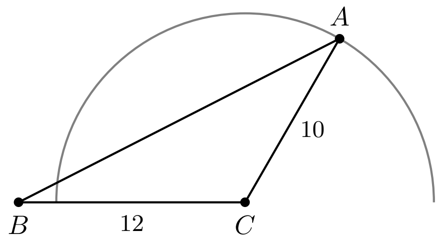
  ii. Theresa places one red tulip at each end of the row. She then places the remaining eight tulips at random between these two red ones. Calculate the probability that the two yellow tulips are next to each other. (1.5 P.)

  iii. Some of the tulip bulbs sold at the shop carry a gene which makes them highly resistant to frost. Of the bulbs producing yellow tulips, 20% carry the gene for frost-resistance; of the bulbs producing the other two colours, 10% carry the gene.

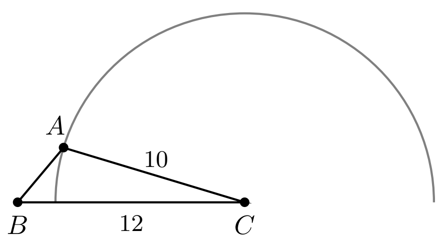
  One week after Theresa has finished arranging her tulips on the balcony, a heavy frost destroys all the tulips except those which carry the gene for frost-resistance. Calculate the probability that exactly two of Theresa's ten tulips survive. (3 P.)

Question 5a: Trigonometry

In this question, we consider triangles with vertices A, B and C, where the two points B and C are 12 units apart and the third point A lies on a semi-circle of radius 10 units centred at C. Depending on the position of the point A, the triangle will change. Examples of two possible triangles ABC are shown below.



Example 1



Example 2

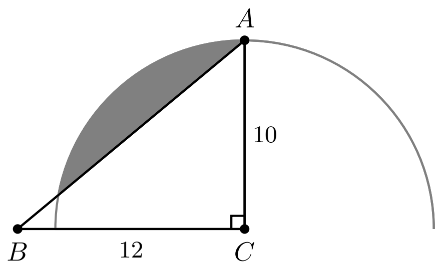
Denote by α, β and γ the interior angles of the triangle at the vertices A, B and C, respectively.

i. In Example 1 shown above, the point A was chosen so that $\gamma = 120°$. Calculate the side-length $\overline{AB}$. (1 P.)

ii. In Example 2, the angle β is equal to $50°$. Calculate the (obtuse) angle α. (1.5 P.)

iii. Now suppose that the point A is free to move along the semi-circle. What is the largest possible value for the angle β? (1 P.)

iv. In Example 3 shown below, $\gamma = 90°$.



Example 3

Find the area of the shaded segment. (2.5 P.)

Question 5b: Exponential Functions

weed	*Unkraut*
pond	*Teich*

Jane notices a weed growing on the surface of her pond. Instead of removing the weed or preventing its growth in any way, Jane decides to study its development over time. She finds that

$$A(t) = \frac{12}{1 + \mathrm{e}^{-0.1t}}$$

where $A(t)$ denotes the area of the pond (in m^2) covered by the weed t days after she first notices it.

i. The surface area of the whole pond is $12\,\mathrm{m}^2$. What percentage of the pond will be covered by the weed 10 days after Jane first notices it? (1 P.)

ii. After how many days will the weed cover $8\,\mathrm{m}^2$ of the pond? Calculate your answer **by hand**. (2.5 P.)

iii. Calculate **by hand** the rate at which the covered area is increasing at time $t = 10$. (2.5 P.)

Best wishes from Thomas Blott, Johannes Börlin, Roman Huber, Andreas Kilberth, Matthieu Penserini, Gérald Prétot, Silke Schewe-Uhlig, Valentina Stauber, Robyn Steiner-Curtis, Raphael Ugolini, Constantin von Weymarn and Alain Zumbiehl.