gymnasiumliestal

Matura 2009 – Written examination in mathematics
Classes: (4BM (Profil M)), 4GL, (4IS), (4IZ), (4S), (4W), (4Wa), (4Wb)

Duration of Exam:	4 hours
Permitted material:	CAS Calculator
	Formula sheets (in English)
	Formula book (in German)
	English-german dictionary
Remarks:	Start each question on a fresh sheet of paper.
	All working done with the calculator should be clearly recorded.
	The marks available for each question - and part-question - are shown.

Calculus (12 marks)

1. Function $f_k(x) = kx^3 - (k+1)x^2$ represents the equation of a family of curves containing the real, positive parameter, k.

   (a) Calculate, **by hand**, the zero points of function f_k. (2 P.)
   (b) Calculate the coordinates of any extreme points and inflexion points on the curve of f_k. (3 P.)
   (c) Calculate, **by hand**, the limiting values of zero points and extreme points of function f_k as a value of k becomes infinitely big. (2 P.)
   (d) Calculate the value for k, so that the graph of function f_k and positive x-axis enclose a region whose area equals $\frac{27}{8}$ units ^{2}. (2 P.)
   (e) For which value of k will the area - of the region described in part (d) - achieve an extreme value? Demonstrate whether this extreme value represents a maximum or a minimum area. (3 P.)

Vector Geometry (12 marks)

2. Points $A(0|5|3)$ and $B(-7|8|3)$ are given, together with the sphere, S, whose equation is: $x^2 + y^2 + z^2 - 12x + 20y + 18z - 72 = 0$.

   (a) Calculate the coordinates of the centre point and the radius of sphere, S. (2 P.)
   (b) The line g, which passes through the points A and B, cuts the sphere S at two points, P_1 and P_2. Find a Cartesian equation for the plane, E_1, which is tangential to the sphere at point P_1. Repeat this task at point P_2. (4 P.)
   (c) Obtain a vector equation for the line of intersection of the two planes E_1 and E_2.
   { If you have been unable to find Cartesian equations for E_1 and E_2 in part (b), you should use the following plane equations:
   $E_1 : 9x + 10y + 13z - 140 = 0$ and $E_2 : 3x + 12y + 12z - 68 = 0$) } (2 P.)
   (d) Decide whether the sphere, S, will, or will not, cut the plane that is the perpendicular bisector of line AB. To obtain full marks for this part-question, a solution, **by-hand**, should be shown. (4 P.)

Probability (12 marks)

3. This question consists of two independent part-questions.

   A gymnasium has 180 students in its 4$^{\text{th}}$ year Classes.

   (a) These 180 students are divided in 3 main groups ('Schwerpunkte'): *Languages*, *Sciences* and *Economics*.
   30 % of the students are in *Economics* groups and 24 of these students are boys. 42 of the students are in the *Science* group and, in this subject, the boys and girls are equal in number. In the *Languages* group, the number of girls is six times as large as the number of boys.

      i. Construct a grid which separates 4$^{\text{th}}$ Class students by their gender (male or female) and by their group (*Languages*, *Sciences* or *Economics*).   (1.5 P.)

     ii. The answer to each of the 3 tasks below should be given in the form of a single, simplified fraction.
   From the 180 students, one student is randomly chosen. The outcomes L, F and E are defined as follows:
   - L: the student chosen is from the *Languages* group.
   - F: the student chosen is female.
   - E: the student chosen is from the *Economics* group.

       A. Calculate the probabilities $P(E)$ and $P(L)$.   (1 P.)

       B. Explain the event $E \cap F$ and calculate $P(E \cap F)$.   (1 P.)

       C. Calculate $P(F|E)$. Explain your results clearly in words.   (1 P.)

   (b) A 3$^{\text{rd}}$ Class student chose to write his 'Maturaarbeit' on the theme of 'Modern Agriculture'. He asked each of the 180 students in the 4$^{\text{th}}$ class to reply to the following two questions:
   - Have you heard of a system of commerce called Fair-Trade?
   - Can you name one label which represents 'certified organic produce'?

   Two-thirds of the students replied "Yes" to the first question - of whom 90 could also name a label of organic produce. (e.g. 'Naturaplan', 'Bio-Engagement', etc.)
   Among those students who had not heard of 'Fair-Trade', 33 were still able to name a label of organic produce.

   Consider the following outcomes, if one of the 180 students is selected at random:
   - OP: the student was able to name a label of organic produce.
   - FT: the student had heard of Fair-Trade.

      i. Show that $P(OP|FT) = 0,75$.   (0.5 P.)

     ii. Construct a tree diagram which represents the information about the outcomes OP and FT, and complete the probabilities of all branches.   (1.5 P.)

     iii. Calculate the probability $P(OP \cap FT)$ and $P(OP \cap \overline{FT})$.   (1 P.)

     iv. One of the results of his research that he printed in his 'Maturaarbeit' was:
   "70 % of the 4$^{\text{th}}$ Class students can name a label for organic produce".
   Is this statement correct? Justify your response.   (1 P.)

Since the number of students is big (180), the probability $P(OP)$ may, if you wish, be considered as constant for the remaining questions.

  v. Calculate the probability that when six randomly-chosen students (from the 4th Class) are questioned, exactly four of them can name a label of organic produce. { If you were unable to answer the previous question, use $P(OP) = 0,7$ } (1.5 P.)

  vi. What is the minimum number of students, n, chosen from the 4th Class that must be asked in order to be $99,5\,\%$ sure that **at least one** of these n students can name a label of organic agriculture? (2 P.)

Calculus (12 marks)

4. The diagram below shows the graph of the continuous function, f, defined for all real numbers. Also shown is the graph of its integral function, F. The zero-points of function, f, are integers.

(a) The graph of the function, f, and the x-axis enclose a region in the 2nd quadrant. Find, **by hand**, the area of its region using only the diagram given. Your final answer should be accurate to one decimal place. (2 P.)
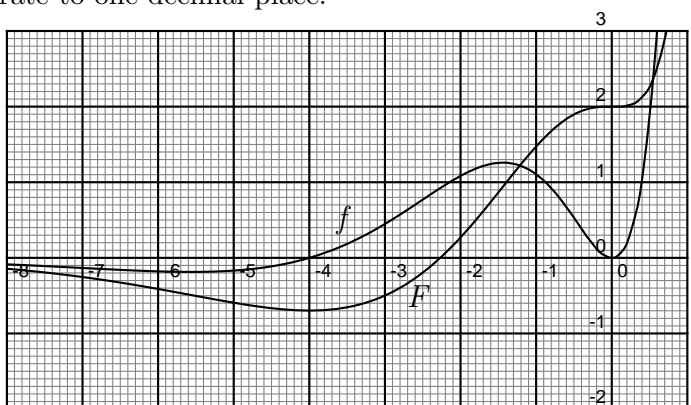
(b) f is a member of the family of curves $f_k(x) = (x^3 + kx^2) \cdot e^x$ where k is a parameter taking all real values. Find the value of k when $f_k(x) = f(x)$. (1.5 P.)

(c) Show, **by hand**, that $F_2(x) = (x^3 - x^2 + 2x - 2) \cdot e^x$ is a possible integral function of $f_2(x)$. (1 P.)

(d) Calculate, **by hand**, the exact area of the region enclosed (in the 2nd quadrant) between the curve of the function $f_2(x)$ and the x-axis. (1.5 P.)

(e) Investigate whether the shape and the position of the graph of function $f_k(x)$ at $x = 0$ are dependent on the value of the parameter, k. (3 P.) { Hint: list all possible cases, using differentiation where necessary }

(f) Find the (positive) value of the parameter, k, so that the graph of function $f_k(x)$ and the x-axis produce two regions with equal areas. (3 P.)

Short Exercises (12 marks)

5. This part consists of three independent questions.

   (a) The lines g_1, g_2 and g_3 form a triangle.

   $$g_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + s \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

   $$g_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

   $$g_3 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + u \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

   Calculate the area of this triangle. (5 P.)

   (b) An initial mass, M_0, increases exponentially at a growth rate of 4,5 % per year, over a period of eight years. At the end of this time, it continues to grow for a further eight years at the new growth rate of p % per year.

   Calculate the value of p, accurate to three significant figures, so that the mass has exactly doubled during these 16 years. (4 P.)

   (c) Calculate the sum af all five-figures numbers containing each of the digits 1, 2, 3, 4 and 5 exactly once. (3 P.)