Gymnasium Liestal Matura 2008

Mathematics (English) **Class 4GL**

Notes: Time allowed is 4 hours.
Begin each exercise on a new sheet of paper.
All your methods must be carefully shown
A list of formulae and a graphic calculator (with no saved data) are permitted
You may use the calculator manual.

Point distribution:

Exercise	1	2	3	4	5	Total
Points	10	12	13	11	12	58

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Exercise 1 - Integration

Note: You may find that the calculator takes several minutes to complete some calculations.

a) A function $f: x \mapsto ax^4 + bx^3 + cx^2 + dx + e$ whose graph is symmetrical about the y axis and goes through the point $P(0|-2)$. At point $Q(6|-\frac{182}{25})$ the curve has a gradient $-\frac{127}{200}$. Find the values of constants a, b, c, d and e. *(3 P)*

b) When the area enclosed by the graph of function f and the x-axis is rotated about the x-axis, a dumbbell shape (see picture) is produced. Find the volume of this object rounded to the nearest whole number. *(3 P)*

(If you have not found an equation for part (a) use the following: $f(x) = \frac{1}{256}x^4 - \frac{593}{1600}x^2 - 2$).

c) When the dumbbells are made with a certain plastic, the mass in grams is the same value as the volume measured in cm^3 (found in part (b)). The dumbbells are to be made lighter by cutting off pieces equally from left and right ends parallel to the y-axis. How long would the dumbbells be if the mass is now 1500 grams (give your answer to 2 dp)? *(2 P)*

Dumbbells may also be constructed by rotating the curves produced by the family of functions g_p where parameter $p > 0$:

$$g_p(x) = p \cdot \left(\left(\frac{x}{4}\right)^4 - \left(\frac{10x}{16}\right)^2 \right) + \frac{x^2 - 100}{50}$$

d) Find the value of parameter p when functions f and g are identical *(1 P)*

e) If dumbbells are constructed by rotating function g about the x-axis what value of p gives a volume 5000 cm^3 (mass 5000 grams)? Give your result to 2 decimal places. *(1 P)*

Exercise 2: Differentiation

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Function $f(x)$ is defined as follows:

$$f(x) = \frac{36x}{x-12} \quad \text{with} \quad x \in \mathbb{R}$$

(see sketch),

a) Find the equations of all asymptotes to the curve and draw these asymptotes on the above sketch. Mark the correct scale on each axis.. *(2.5 P)*

b) Sketch the graph of $f'(x)$ on the coordinate system below. *(1 P)*

Point $P(P_x|P_y)$ has an x-coordinate value greater than 12, and lies on the curve of function f. Point O is the origin and point Q, which lies on the x-axis, has the same x-coordinate as point P.

c) Find an expression (in terms of x) for the area function $A(x)$ that represents the area of of triangle OPQ. *(1 P)*

d) Find the coordinates of the point P which produce a triangle OPQ of minimum or maximum area and calculate this area. *(2.5 P)*

Point $B(12|36)$ is given. A line with gradient 1 passes through point B and cuts the graph of f at point P_1 and P_2.

e) Find the coordinates of P_1 and P_2. *(2 P)*

f) Show that the lengths of $\overline{BP_1}$ and $\overline{BP_2}$ are the same. *(1 P)*

g) Explain why the distance between $\overline{P_1P_2}$ represents the shortest distance between the two parts of the curve of f. *(1 P)*

h) Find all whole number values for x which satisfy both of the following conditions $f(x) < 100$ and $f''(x) > 1$ *(1 P)*

Exercise 3 - Vector Geometry

The Spaceship *Enterprise*, sets off from Galaxy M104 along a straight path containing the points $A(0|4|-2)$ and $B(-5|-7|-6)$. Point B is reached after point A. A small meteorite approaches the spaceship along a path given by the straight line $g_{\text{meteorite}}$ defined by:

$$g_{\text{meteorite}}: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} .$$

a) Treating both the spaceship and the meteorite as points, show whether a collision would occur between them. (The 'SOLVE' operator of your calculator must not be used here.) *(2 P)*

As a precaution, Captain Kirk reprograms the path of the spaceship so that it now travels along the path $g_{\text{Enterprise}}$ given by:

$$g_{\text{Enterprise}}: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -7 \\ -6 \end{pmatrix} + s \cdot \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} .$$

b) What is the angle α between the old path and the new path of the spaceship? *(2 P)*

c) Scientific Officer Mr. Spock calculates the cartesian equation of the plane which contains both flight paths of the *Enterprise*. What equation does he find? *(2 P)*

The inhabitants of planet Krrgh, located at $K(8|16|12)$, feel threatened. They send a destructive laser beam to strike the spaceship at the point Q on its path $g_{\text{Enterprise}}$. The point Q has been chosen so that the laser beam has the shortest distance to travel in order to meet the path of the (*Enterprise*).

d) What are the coordinates of point Q ? *(3 P)*

The *Enterprise* reacts by constructing a spherical protective shield around itself. The spherical shield has centre $M(7|13|2)$ which is the current position of the spaceship. At that instant, a laser beam strikes the protective shield at point $S(9|17|8)$ and is reflected by the plane P_S tangent to the sphere at point S. The cartesian equation of plane P_S is:

$$P_S: \quad x + 2y + 3z - 67 = 0 .$$

As the laser beam strikes the protective shell, the surface of the shell glows bright blue.

e) Calculate the surface area of the blue shell, rounded to the nearest whole number *(1.5 P)*

f) Find the direction vector $\vec{e}$ of the reflected beam, where $\vec{e}$ contains only whole numbers. *(2.5 P)*

Exercise 4 - Combinations and Probability

A teacher of a class of 18 boys and 14 girls is given 25 tickets for a football final.

a) How many different groups of 25 can be formed from these students if there are no restrictions? *(1 P)*

b) How many groups of 25 students could be formed if exactly 10 must be girls? *(1.5 P)*

The teacher decides to distribute the tickets to her class by lottery. She places 25 slips with the word Win and 7 blank slips into a deep urn and asks each student in turn to select a slip. Heidi is the second student to choose a slip and she complains that her probability of winning is less than Lena's, the first student.

c) Show, with the aid of a calculation, that Heidi's claim is incorrect. *(1.5 P)*

The 25 ticket winners arrive several hours early and find in the stadium area a goal practice wall with an upper and a lower hole. Renno is an experienced goal shooter and when he chooses the upper hole scores a goal with a probability of 0,1. When he chooses the lower hole, he scores a goal 8 out of 20 times on average.

d) Renno has two kicks, aiming first at the lower hole then the upper hole. With what probability does he score exactly one goal? *(1 P)*

e) If he scores a goal with his first kick at the lower hole, what is the probability that his kick at the upper hole is also successful? *(1 P)*

f) Renno now only aims at the lower hole. How many times must he kick in order to score at least one goal with a probability of 80%? *(2 P)*

g) With what probability would Renno score at least 3 goals out of 15 shots at the upper hole? *(2 P)*

Immediately before the game begins, an announcement is made by the stadium's director that football fever has taken hold in the town and 80% of the town's population support a plan to extend the stadium. At half-time, the students decide to test this statement. They decide to ask 131 people randomly selected people in the stadium whether they support the plan to extend the stadium

h) Give two brief comments on the method the students have chosen to test the statement made by the stadium's director. *(1 P)*

Exercise 5a - Exponential Functions

Measurements of atmospheric CO_2 concentrations have shown exponential increase in recent years. One model, Model A (`http://metoffice.gov.uk`) predicts that CO_2 concentrations will increase from 300ppmv at the end of 1980 to double this level by the end of 2010, according to the formula

$$C(t) = a \cdot \mathrm{e}^{kt}$$

where C is the concentration of CO_2, e is Euler's number t time in years from the end of 1980 , a and k are constants.

a) What are the values of a and k in Model A? *(1 P)*

b) Calculate the yearly rate of change of CO_2 concentration predicted by Model A at the end of 2010. *(1 P)*

If you have not found values for part a) use $a = 290$ *et* $k = 0.029$.

Other researachers propose a Model B where atmospheric CO_2 increases from the end of 1980 as follows:

$$\ln(C(t)) = \ln(250) + 0.04t$$

where C is concentration of CO_2 in ppmv and t time in years after the end of 1980.

c) According to model B what was the CO_2 concentration in 1980? *(0.5 P)*

d) According to Model B in which year would the concentration of CO_2 have risen to double the end of 1980 level? *(1 P)*

e) In which year do both models predict the same CO_2 concentration? *(1.5 P)*

f) If ecological measures to cut atmospheric CO_2 by 2% per year begin to take effect when CO_2 concentration reaches 800 ppmv, in how many years after the concentration reaches 800 ppmv would the atmospheric CO_2 level reach 300 ppmv again? *(1 P)*

Exercise 5b - Trigonometry

a) The 7 points $P_1, P_2, \ldots$ are equally spaced anti-clockwise points on a circle of radius 14 mm and centre point C. A property of circles is that angle alpha is exactly twice the size of angle beta.

a) Find the exact value of the following ratio:

$$\frac{\text{length of segment } P_5P_1}{\text{length of segment } CP_1}.$$

(1.5 P)

b) The shape of a British 50-pence piece is constructed in the following manner:

- 7 equally spaced points lie on a circle of radius 14 mm.
- Each of the seven curved edges (*Kante*) is the arc of a circle(see diagram). The first arc goes from P_1 to P_2 (with P_5 as the centre), the second goes from P_2 to P_3 (with P_6 as the centre), the third goes from P_3 to P_4 (with P_7 as the centre), etc...

(i) Calculate the area of this geometrical shape, accurate to the nearest mm^2 *(3 P)*

Advice : If you have not found a value for part(a), use 1.8 *as your value for the ratio of the line segments.*

(ii) The coin has a thickness of exactly 2 mm. Calculate the total surface area of such a coin. Give your result to the nearest mm^2 *(1,5 P)*