Mathematik

Klassen 4FIS/4GL/4MS/4SZ/4Wa/4Wb/5KSW

Instructions: The length of this exam is 4 hours.
Begin each question on a separate sheet of paper.

Materials available: Graphic calculator and Formula Sheet(in english). No dictionaries.

Point Distribution:

Question	1	2	3	4	5	6	Total
Points	8	5	10	12	10	5	50

Question 1: Analysis

A fully rational function, called f, has the following properties:

(i) $f(4) = 2$

(ii) $f'(4) = 0$

(iii) $f''(x) < 0$ for all $x \in R$

(iv) $\int_0^6 f(x)dx = 0$

Give **full and precise** explanations as your answers to the following questions:

a) How many points of inflexion does f have? (1P.)

b) How many stationary points does f have? (2P.)

c) How many zero-values does f have? (2P.)

d) Show the function f which satisfies all the conditions from (i) to (iv). (3P.)

Question 2: Decay

A chemical substance is heated. At time $t = 0$, the substance begins to burn. It has an initial mass of $m(0)$ and, at any later time t, its mass, called $m(t)$, can be found using the following rule:

$$m(t) = m(0) \cdot \frac{2e^t}{1 + e^{2t}} \qquad (t \text{ in minutes})$$

In this question, give all your numerical answers accurate to one decimal place.

a) After one minute, what percentage of the substance's mass has been burnt? (1P.)

b) After how many minutes has 99.9 % of the mass of the substance burnt away? (1P.)

By differentiating the function $m(t)$, we obtain a rule for the burning-rate of the mass.

c) What is the burning-rate at time $t = 0$? (1P.)

d) At what time will the burning-rate reach its maximum value? (1P.)

e) Compare your results to parts (c) and (d) with those that you would expect from a process of 'Exponential Decay'. (1P.)

Question 3: Logarithmic Function

a) A family of curves are given by the following equation

$$y = f_a(x) = \frac{k \cdot \ln(x+1)}{(x+1)^2} \qquad \text{for } x \geqslant 0 \text{ and } k > 0.$$

a1) Sketch the appropriate curve from this family for $k = 10$.

a2) Show that all curves in this family have the same x-value at their stationary point. What is the y-value of this stationary point?

a3) Find the value for the constant k, so that the area A_a of the unending region R_a, which lies between the curve of the function $f_a(x)$ and the x-axis, has the value 9.

a4) {The value $k = 9$ should be used for the following task}
What is the volume V_a, when the region R_a is rotated 360 degrees about the x-axis?

(4P.)

Parts b) and c) of this question may be solved independently of each other.
For these parts, the value $k = 9$ still applies.

b) Find the values of the constants p and q in the function whose equation is:

$$y = f_b(x) = \frac{p \cdot q^3 \cdot x}{(x+q)^3} \qquad \text{for } x \geqslant 0 \text{ and } q > 0$$

- so that the curves of both $f_b(x)$ and $f_a(x)$ have the same gradient at $x = 0$, and
- that the unending regions R_b and R_a have the same size (i.e. $A_b = A_a$).

(3P.)

c) Given the following function with equation:

$$y = f_c(x) = \frac{5x}{(x+1)^2} \qquad \text{for } x \geqslant 0.$$

c1) Calculate the area A_c of the unending region R_c which lies between the curve of $f_c(x)$ and the x-axis.

c2) What is the volume V_c, when the region R_c is rotated 360 degrees about the x-axis?

c3) Obtain the integral functions $I_a(x) = \int_0^x f_a(t)dt$ and $I_c(x) = \int_0^x f_c(t)dt$ and use these to explain the result for A_c as $x \to \infty$.

(3P.)

Question 4: Vector Geometry

a) Plane $\epsilon : x + 2y + 2z = 12$ and points $P(6 \mid 1 \mid 2)$ and $F_1(12 \mid 5 \mid 4)$ are given.

a1) Sketch, using oblique axes, the plane ϵ. (1P.)

a2) Show that the point P lies in plane ϵ and indicate its position on your diagram. (1P.)

a3) Find the coordinates of the point S, in plane ϵ, which is closest to the point F_1. (2P.)
{ If you have not found an answer, use $S(8 \mid 2 \mid 0)$ for the remainder of the question }

b) The walls, ceiling and floor of a large cube-shaped room are defined by the following three pairs of parallel planes:

$$x = 0 \text{ und } x = 12 \qquad y = 0 \text{ und } y = 12 \qquad z = 0 \text{ und } z = 12.$$

The room contains a spider's web which lies in the plane ϵ. (All units are in metres.)

b1) The two flies which are in the room are unable to see each other due to the presence of a spider sitting in the web at point P. If the first fly is sitting at point F_1, and the second fly is sitting on a wall at point $F_2(a \mid 0 \mid b)$, find the values of a and b. (2P.)
{ If you have not found an answer, use $F_2(3.6 \mid 0 \mid 3.2)$ for the rest of the question }

b2) The spider notices the fly at F_1 and moves across his web to point S. From this position, the spider can now just see both flies in his range of vision. To the nearest degree, what is his range of vision? (1P.)

b3) The fly at F_1 appears to be asleep. The spider, who now sits at point S, estimates that if his surface distance - using only his net, the walls and the floor - from F_1 is less than 7.5 metres, he will catch the fly before it wakes up. If this reasoning is correct, can the fly escape? Justify your answer by showing any necessary calculations. (2P.)

b4) Before the spider has made his decision, the fly - who was not asleep! - sees a hole in the web at point H, and sets off directly towards it, on a straight flight-path, in the direction of the vector

$$\vec{s} = \begin{pmatrix} -8 \\ -4 \\ -1 \end{pmatrix}.$$

Find the coordinates of the point H, and find also the coordinates of point N (a point on the fly's path between F_1 and H), where the fly passes closest to the hungry spider. (3P.)

Question 5: Probability

The electronic post (E-mail) is one of the most important means of communication for modern business. Unfortunately, unwelcome publicity - usually called Spam - represents a large part of the total E-mail traffic.

In order to protect users from receiving floods of Spam, all the main E-mail providers employ a Spam-Filter system. These filters examine each incoming E-mail and decide whether it is genuine, and therefore welcome, or that is a Spam. Those E-mails which have been classified as genuine are directed to the 'Inbox' of the addressee, whereas those messages which are judged to be Spam are placed in a special 'Spam-File'.

a) Until recently, a method which was commonly used by the senders of Spam, in order to outwit these Spam-Filters, was to randomly mix the letters of a 'genuine' word in the text. Calculate how many **different** arrangements of letters could be produced by applying this method to the word "'`assessment`"'. (1.5P.)

b) Anna finds 15 E-mails (all with attachments) in her Spam-File. Four of these contain a virus which would spread throughout her computer if the attachment were opened.
Anna opens, at random, the attachments of three of the E-mails in her Spam-File. What is the probability that Anna releases at least one virus into her computer? (1.5P.)

According to statistics dating from February 2007, Spams represent 81 % of all E-mail traffic and only 19 % of E-mails are genuine. Clearly, the more Spams that the filter can detect and the fewer genuine E-mails that it rejects is a measure of the efficiency of the Spam-Filter.
At present, the best Span-Filter is able to correctly recognise 99.97 % of all Spams as such, and to reject(in error) only 0.025 % of all genuine E-mails.

c) A large company, using one of these filters, receives 250'000 E-mails per day. How many incoming E-mails which are genuine would the company expect to 'lose' (i.e. be rejected) each day? (2P.)

d) An employee of the company finds an E-mail in his Inbox. What is the probability that it is a Spam? {Construct a tree diagram to illustrate your answer.} (3P.)

In the case that you are unable to obtain solve part (d), you may still continue by using the probability that an E-mail found in the employee's Inbox is a Spam has a value of 0.128 %.

e) One of these employees has 200 E-mails in his Inbox. What is the probability that exactly three of these are Spams? (2P.)

Question 6: Arithmetic

There exists only one ten-figure number with all the following properties:

- Each of the ten digits from 0 to 9 appear.
- For $n = 1, 2, 3, 4, \ldots, 9, 10$, the first n digits of this ten-figure number show a number which is exactly divisible by n.

We wish to find this number!

False example : using the ten-figure number 1234567890, we obtain the working stages

	1	is divisible by 1 ? yes,
	12	is divisible by 2 ? yes,
	123	is divisible by 3 ? yes,
but	1234	is divisible by 4 ? no.

Only one of the following suggestions (a) to (f) will lead to the ten-figure number that we require. Show, for each of the false suggestions, the condition that is not satisfied. Finally, build up the complete ten-figure number that we are looking for.

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(a)     3 0 9 6 _ _ _ _ _ _

(b)     3 _ 5 6 _ _ _ _ _ 0

(c)     3 _ _ 6 _ 4 7 _ _ _

(d)     3 8 _ 9 5 _ _ _ _ _

(e)     _ 8 _ _ _ 4 7 6 _ _

(f)     _ 8 _ 6 _ _ 7 _ 1 _
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(5P.)

With best wishes for your success, from: Maria Montero, Thomas Blott, Bernhard Felder, Andreas Immeli, Guido Lafranchi, Eric Lucas, Mic Rasmussen.